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# Real hypersurfaces in the complex quadric with parallel structure Jacobi operator ☆

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ABSTRACT

First we introduce the notion of parallel structure Jacobi operator for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. Next we give a complete classification of real hypersurfaces in $Q^m = SO_{m+2}/SO_mSO_2$ with parallel structure Jacobi operator.



## 1. Introduction

In case of Hermitian symmetric spaces of rank 2, usually we can give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [1,2,15,16]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure $J$ and the quaternionic Kähler structure $\mathfrak{J}$ on $SU_{2,m}/S(U_2U_m)$. The rank of $SU_{2,m}/S(U_2U_m)$ is 2 and there are exactly two types of singular tangent vectors $X$ of $SU_{2,m}/S(U_2U_m)$ which are characterized by the geometric properties $JX \in \mathfrak{J}X$ and $JX \perp \mathfrak{J}X$ respectively.

As another kind of Hermitian symmetric space with rank 2 of compact type different from the above ones, we can give the example of complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, which is a complex hypersurface in complex projective space $\mathbb{C}P^m$ (see Berndt and Suh [3], and Smyth [14]). The complex quadric also can be

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regarded as a kind of real Grassmann manifold of compact type with rank 2 (see Kobayashi and Nomizu [7]). Accordingly, the complex quadric admits both a complex conjugation structure $A$ and a Kähler structure $J$, which anti-commutes with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple $(Q^m, J, g)$ is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [6] and Reckziegel [12]).

In addition to the complex structure $J$ there is another distinguished geometric structure on $Q^m$, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an $S^1$-bundle of real structures, that is, complex conjugations $A$ on the tangent spaces of $Q^m$. The set is denoted by $\mathfrak{A}_{[z]} = \{A_{\lambda\bar{z}} | \lambda \in S^1 \subset \mathbb{C}\}$, $[z] \in Q^m$, and it is the set of all complex conjugations defined on $Q^m$. Then $\mathfrak{A}_{[z]}$ becomes a parallel rank 2-subbundle of End $TQ^m$. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle $TM$ of a real hypersurface $M$ in $Q^m$. Here the notion of parallel vector bundle $\mathfrak{A}$ means that $(\bar{\nabla}_X A)Y = q(X)JAY$ for any vector fields $X$ and $Y$ on $Q^m$, where $\bar{\nabla}$ and $q$ denote a connection and a certain 1-form defined on $T_{[z]}Q^m$, $[z] \in Q^m$ respectively (see Smyth [14]).

Recall that a nonzero tangent vector $W \in T_z Q^m$ is called singular if it is tangent to more than one maximal flat in $Q^m$. There are two types of singular tangent vectors for the complex quadric $Q^m$:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then $W$ is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal, where $V(A)$ denotes the $(+1)$-eigenspace and $JV(A)$ the $(-1)$-eigenspace of the conjugation $A$.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then $W$ is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

On the other hand, Okumura [13] proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_1U_m)$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^m$ for some $k \in \{0, \ldots, m-1\}$. For the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$ a classification was obtained by Berndt and Suh [14]. The Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$. Moreover, in [16] we have proved that the Reeb flow on a real hypersurface in $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$ is isometric if and only if $M$ is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$. For the complex quadric $Q^m = SO_{m+2}/SO_2SO_m$, Berndt and Suh [1] have obtained the following result:

**Theorem A.** *Let $M$ be a real hypersurface of the complex quadric $Q^m$, $m \geq 3$. Then the Reeb flow on $M$ is isometric if and only if $m$ is even, say $m = 2k$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

On the other hand, Jacobi fields along geodesics of a given Riemannian manifold $(M, g)$ satisfy a well known differential equation. This equation naturally inspires the so-called Jacobi operator. That is, if $R$ denotes the curvature operator of $M$, and $X$ is tangent vector field to $M$, then the Jacobi operator $R_X \in End(T_xM)$ with respect to $X$ at $x \in M$, defined by $(R_XY)(x) = (R(Y,X)X)(x)$ for any $X \in T_xM$, becomes a self adjoint endomorphism of the tangent bundle $TM$ of $M$. Thus, each tangent vector field $X$ to $M$ provides a Jacobi operator $R_X$ with respect to $X$. In particular, for the Reeb vector field $\xi$, the Jacobi operator $R_\xi$ is said to be a *structure Jacobi operator*.

Recently Ki, Pérez, Santos and Suh [8] have investigated the Reeb parallel structure Jacobi operator in the complex space form $M_m(c)$, $c \neq 0$ and have used it to study some principal curvatures for a tube over a totally geodesic submanifold. In particular, Pérez, Jeong and Suh [11] have investigated real hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$ with parallel structure Jacobi operator, that is, $\nabla_X R_\xi = 0$ for any tangent vector field $X$ on $M$. Jeong, Suh and Woo [5] and Pérez and Santos [9] have generalized such a notion to the recurrent structure Jacobi operator, that is, $(\nabla_X R_\xi)Y = \beta(X)R_\xi Y$ for a certain 1-form $\beta$ and any vector fields $X, Y$

on $M$ in $G_2(\mathbb{C}^{m+2})$. Moreover, Pérez, Santos and Suh [10] have further investigated the property of the Lie $\xi$-parallel structure Jacobi operator in complex projective space $\mathbb{C}P^m$, that is, $\mathcal{L}_\xi R_\xi = 0$.

When we consider a hypersurface $M$ in the complex quadric $Q^m$, the unit normal vector field $N$ of $M$ in $Q^m$ can be divided into two cases: $N$ is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [3,4,17,18]). In the first case where $M$ has an $\mathfrak{A}$-isotropic unit normal $N$, we have asserted in [3] that $M$ is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in $Q^{2k}$. In the second case when $N$ is $\mathfrak{A}$-principal we have proved that $M$ is locally congruent to a tube over a totally geodesic and totally real submanifold $S^m$ in $Q^m$ (see [4]).

In this paper we consider the case when the structure Jacobi operator $R_\xi$ of $M$ in $Q^m$ is parallel, that is, $\nabla_X R_\xi = 0$ for any tangent vector field $X$ on $M$, and first we prove the following

**Main Theorem 1.** *Let $M$ be a Hopf real hypersurface in $Q^m$, $m{\geq}3$, with parallel structure Jacobi operator. Then the unit normal vector field $N$ is singular, that is, $N$ is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

On the other hand, in [20] we have considered the notion of parallel normal Jacobi operator $\bar{R}_N$ for a real hypersurface $M$ in $Q^m$, that is, $\nabla_X \bar{R}_N = 0$ for any tangent vector fields $X$ and a unit normal vector field $N$ on $M$, and have proved a non-existence property, where the normal Jacobi operator $\bar{R}_N$ is defined by $\bar{R}_N X = \bar{R}(X,N)N$ from the curvature tensor $\bar{R}$ of the complex quadric $Q^m$. Motivated by this result, and using Theorem A and Main Theorem 1, we give another non-existence property for Hopf real hypersurfaces in $Q^m$ with parallel structure Jacobi operator as follows:

**Main Theorem 2.** *There do not exist any Hopf real hypersurfaces in $Q^m$, $m{\geq}3$ with parallel structure Jacobi operator.*

## 2. The complex quadric

For more background to this section we refer to [3,4,6,7,12,17,18]. The complex quadric $Q^m$ is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_1^2 + \cdots + z_{m+2}^2 = 0$, where $z_1, \ldots, z_{m+2}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip $Q^m$ with the Riemannian metric which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure $(J, g)$ on the complex quadric. For each $z \in Q^m$ we identify $T_z\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus \mathbb{C}z$ of $\mathbb{C}z$ in $\mathbb{C}^{m+2}$ (see Kobayashi and Nomizu [7]). The tangent space $T_zQ^m$ can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus (\mathbb{C}z \oplus \mathbb{C}\rho)$ of $\mathbb{C}z \oplus \mathbb{C}\rho$ in $\mathbb{C}^{m+2}$, where $\rho \in \nu_z Q^m$ is a normal vector of $Q^m$ in $\mathbb{C}P^{m+1}$ at the point $z$.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group $SU_{m+2}$, namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o = [0, \ldots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing $o$ is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric $Q^m$ as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of $Q^m$ as a Hermitian symmetric space of rank 2. The complex quadric $Q^1$ is isometric to a sphere $S^2$ with constant curvature, and $Q^2$ is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on.

For a unit normal vector $\rho$ of $Q^m$ at a point $z \in Q^m$ we denote by $A = A_\rho$ the shape operator of $Q^m$ in $\mathbb{C}P^{m+1}$ with respect to $\rho$. The shape operator is an involution on the tangent space $T_zQ^m$ and

$$T_zQ^m = V(A_\rho) \oplus JV(A_\rho),$$

where $V(A_\rho)$ is the +1-eigenspace and $JV(A_\rho)$ is the $(-1)$-eigenspace of $A_\rho$. Geometrically this means that the shape operator $A_\rho$ defines a real structure on the complex vector space $T_zQ^m$, or equivalently, is a complex conjugation on $T_zQ^m$. Since the real codimension of $Q^m$ in $\mathbb{C}P^{m+1}$ is 2, this induces an $S^1$-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations.

There is a geometric interpretation of these conjugations. The complex quadric $Q^m$ can be viewed as the complexification of the $m$-dimensional sphere $S^m$. Through each point $z \in Q^m$ there exists a one-parameter family of real forms of $Q^m$ which are isometric to the sphere $S^m$. These real forms are congruent to each other under action of the center $SO_2$ of the isotropy subgroup of $SO_{m+2}$ at $z$. The isometric reflection of $Q^m$ in such a real form $S^m$ is an isometry, and the differential at $z$ of such a reflection is a conjugation on $T_zQ^m$. In this way the family $\mathfrak{A}$ of conjugations on $T_zQ^m$ corresponds to the family of real forms $S^m$ of $Q^m$ containing $z$, and the subspaces $V(A) \subset T_zQ^m$ correspond to the tangent spaces $T_zS^m$ of the real forms $S^m$ of $Q^m$.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of $Q^m$ can be described in terms of the complex structure $J$ and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned}\bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY.\end{aligned}$$

Then from the equation of Gauss the curvature tensor $R$ of $M$ in complex quadric $Q^m$ is defined so that

$$\begin{aligned}R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ &\quad + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY \\ &\quad + g(SY,Z)SX - g(SX,Z)SY,\end{aligned}$$

where $S$ denotes the shape operator of $M$ in $Q^m$.

For every unit tangent vector $W \in T_zQ^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. If $0 < t < \pi/4$ then the unique maximal flat containing $W$ is $\mathbb{R}X \oplus \mathbb{R}JY$. Later we will need the eigenvalues and eigenspaces of the Jacobi operator $R_W = R(\cdot, W)W$ for a singular unit tangent vector $W$.

1. If $W$ is an $\mathfrak{A}$-principal singular unit tangent vector with respect to $A \in \mathfrak{A}$, then the eigenvalues of $R_W$ are 0 and 2 and the corresponding eigenspaces are $\mathbb{R}W \oplus J(V(A) \ominus \mathbb{R}W)$ and $(V(A) \ominus \mathbb{R}W) \oplus \mathbb{R}JW$, respectively.
2. If $W$ is an $\mathfrak{A}$-isotropic singular unit tangent vector with respect to $A \in \mathfrak{A}$ and $X, Y \in V(A)$, then the eigenvalues of $R_W$ are 0, 1 and 4 and the corresponding eigenspaces are $\mathbb{R}W \oplus \mathbb{C}(JX + Y)$, $T_zQ^m \ominus (\mathbb{C}X \oplus \mathbb{C}Y)$ and $\mathbb{R}JW$, respectively.

## 3. Some general equations

Let $M$ be a real hypersurface in $Q^m$ and denote by $(\phi, \xi, \eta, g)$ the induced almost contact metric structure. Note that $\xi = -JN$, where $N$ is a (local) unit normal vector field of $M$. The tangent bundle $TM$ of $M$ splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of $TM$. The structure tensor field $\phi$ restricted to $\mathcal{C}$ coincides with the complex structure $J$ restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define the maximal $\mathfrak{A}$-invariant subspace of $T_zM$, $z\in M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

**Lemma 3.1.** *([17]) For each $z \in M$ we have:*

(i) *If $N_z$ is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*
(ii) *IF $N_z$ is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

We now assume that $M$ is a Hopf hypersurface. Then the shape operator $S$ of $M$ in $Q^m$ satisfies

$$S\xi = \alpha\xi$$

with the Reeb function $\alpha = g(S\xi, \xi)$ on $M$. When we consider a transform $JX$ of the Kaehler structure $J$ on $Q^m$ for any vector field $X$ on $M$ in $Q^m$, we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal $N$ to $M$. We now consider the Codazzi equation

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) = \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned}$$

Putting $Z = \xi$ we get

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = -2g(\phi X, Y) \\ + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ &= g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

Reinserting this into the previous equation yields

$$\begin{aligned} & g((\nabla_X S)Y - (\nabla_Y S)X, \xi) \\ &= -2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ &\quad + 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X) \\ &\quad + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Altogether this implies

$$\begin{aligned}0 &= 2g(S\phi SX,Y) - \alpha g((\phi S + S\phi)X,Y) - 2g(\phi X,Y)\\ &\quad + g(X,AN)g(Y,A\xi) - g(Y,AN)g(X,A\xi)\\ &\quad - g(X,A\xi)g(JY,A\xi) + g(Y,A\xi)g(JX,A\xi)\\ &\quad + 2g(\xi,AN)g(X,A\xi)\eta(Y) - 2g(X,AN)g(\xi,A\xi)\eta(Y)\\ &\quad - 2g(\xi,AN)g(Y,A\xi)\eta(X) + 2g(Y,AN)g(\xi,A\xi)\eta(X).\end{aligned}$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \le t \le \frac{\pi}{4}$ (see Proposition 3 in [12]). Note that $t$ is a function on $M$. First of all, since $\xi = -JN$, we have

$$\begin{aligned}N &= \cos(t)Z_1 + \sin(t)JZ_2,\\ AN &= \cos(t)Z_1 - \sin(t)JZ_2,\\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1,\\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1.\end{aligned}$$

This implies $g(\xi, AN) = 0$ and hence

$$\begin{aligned}0 &= 2g(S\phi SX,Y) - \alpha g((\phi S + S\phi)X,Y) - 2g(\phi X,Y)\\ &\quad + g(X,AN)g(Y,A\xi) - g(Y,AN)g(X,A\xi)\\ &\quad - g(X,A\xi)g(JY,A\xi) + g(Y,A\xi)g(JX,A\xi)\\ &\quad - 2g(X,AN)g(\xi,A\xi)\eta(Y) + 2g(Y,AN)g(\xi,A\xi)\eta(X).\end{aligned}$$

The curvature tensor $R(X,Y)Z$ for a real hypersurface $M$ in $Q^m$ is given by

$$\begin{aligned}R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X\\ &\quad - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\\ &\quad + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX\\ &\quad - g(JAX,Z)JAY + g(SY,Z)SX - g(SX,Z)SY.\end{aligned}$$

From this, putting $Y = Z = \xi$ and using $g(A\xi, N) = 0$, a structure Jacobi operator is defined by

$$\begin{aligned}R_\xi(X) &= R(X,\xi)\xi\\ &= X - \eta(X)\xi + g(A\xi,\xi)AX - g(AX,\xi)A\xi\\ &\quad - g(JAX,\xi)JA\xi + g(S\xi,\xi)SX - g(SX,\xi)S\xi.\end{aligned}$$

Hereafter, we will apply the following lemmas which will be useful to prove our results in the introduction.

**Lemma 3.2.** ([17]) *Let $M$ be a Hopf hypersurface in $Q^m$ such that the normal vector field $N$ is $\mathfrak{A}$-principal everywhere. Then $\alpha$ is constant. Moreover, if $X \in \mathcal{C}$ is a principal curvature vector of $M$ with principal curvature $\lambda$, then $2\lambda \ne \alpha$ and $\phi X$ is a principal curvature vector of $M$ with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

**Lemma 3.3.** ([17]) *Let $M$ be a Hopf hypersurface in $Q^m$, $m \geq 3$, such that the normal vector field $N$ is $\mathfrak{A}$-isotropic everywhere. Then $\alpha$ is constant.*

## 4. A key lemma

The curvature tensor $R(X,Y)Z$ for a Hopf real hypersurface $M$ in $Q^m$ induced from the curvature tensor of $Q^m$ is given in section 3. Now the structure Jacobi operator $R_\xi$ from section 3 can be rewritten as follows:

$$\begin{aligned} R_\xi(X) &= R(X,\xi)\xi \\ &= X - \eta(X)\xi + \beta AX - g(AX,\xi)A\xi - g(AX,N)AN \\ &\quad + \alpha SX - g(SX,\xi)S\xi, \end{aligned} \tag{4.1}$$

where we have put $\alpha = g(S\xi,\xi)$ and $\beta = g(A\xi,\xi)$, because we assume that $M$ is Hopf. The Reeb vector field $\xi = -JN$ and the anti-commuting property $AJ = -JA$ gives that the function $\beta$ becomes $\beta = -g(AN,N)$. When this function $\beta = g(A\xi,\xi)$ identically vanishes, we say that a real hypersurface $M$ in $Q^m$ is $\mathfrak{A}$-isotropic as in section 1.

Here we use the assumption of being parallel structure Jacobi operator, that is, $\nabla_Y R_\xi = 0$. Then (4.1) gives that

$$\begin{aligned} 0 &= \nabla_Y R_\xi(X) = \nabla_Y(R_\xi(X)) - R_\xi(\nabla_Y X) \\ &= -(\nabla_Y\eta)(X)\xi - \eta(X)\nabla_Y\xi + (Y\beta)AX \\ &\quad + \beta\{\bar{\nabla}_Y(AX) - A\nabla_Y X\} - g(X,\bar{\nabla}_Y(A\xi))A\xi \\ &\quad - g(X,A\xi)\bar{\nabla}_Y(A\xi) - g(X,\bar{\nabla}_Y(AN))AN - g(X,AN)\bar{\nabla}_Y(AN) \\ &\quad + (Y\alpha)SX + \alpha(\nabla_Y S)X - Y(\alpha^2)\eta(X)\xi \\ &\quad - \alpha^2(\nabla_Y\eta)(X)\xi - \alpha^2\eta(X)\nabla_Y\xi \\ &= -g(\phi SY,X)\xi - \eta(X)\phi SY + (Y\beta)AX \\ &\quad + \beta\{q(Y)JAX + g(SX,Y)AN\} \\ &\quad - g(X,q(Y)JA\xi + A\phi SY + \alpha\eta(Y)AN)A\xi \\ &\quad - g(X,A\xi)\{q(Y)JA\xi + A\phi SY + \alpha\eta(Y)AN\} \\ &\quad - g(X,q(Y)JAN - ASY)AN - g(X,AN)\{q(Y)JAN - ASY\} \\ &\quad + (Y\alpha)SX + \alpha(\nabla_Y S)X - Y(\alpha^2)\eta(X)\xi \\ &\quad - \alpha^2(\nabla_Y\eta)(X)\xi - \alpha^2\eta(X)\nabla_Y\xi, \end{aligned} \tag{4.2}$$

where we have used the following formulae

$$\begin{aligned} \bar{\nabla}_Y(A\xi) &= (\bar{\nabla}_Y A)\xi + A(\bar{\nabla}_Y\xi) \\ &= q(Y)JA\xi + A\phi SY + g(SY,\xi)AN, \\ \bar{\nabla}_Y(AN) &= (\bar{\nabla}_Y A)N + A\bar{\nabla}N = q(Y)JAN - ASY, \end{aligned}$$

and

$$\bar{\nabla}_Y(AX) = (\bar{\nabla}_Y A)X + A\bar{\nabla}_Y X = q(Y)JAX + A(\nabla_Y X + \sigma(X,Y)).$$

From this, by taking the inner product of (4.2) with the unit normal $N$, we have

$$\begin{aligned} 0 = &(Y\beta)g(AX,N) + \beta\{q(Y)g(JAX,N) + g(SX,Y)g(AN,N)\} \\ &- g(X,A\xi)q(Y)g(JA\xi,N) \\ &- g(X,A\xi)\{g(A\phi SY,N) + \alpha\eta(Y)g(AN,N)\} \\ &- g(X,q(Y)JAN - ASY)g(AN,N) \\ &- q(Y)g(JAN,N)g(X,AN) + g(X,AN)g(ASY,N) \end{aligned} \tag{4.3}$$

Then, first, by putting $X = \xi$ and using $g(A\xi,N) = 0$, we have

$$-\beta g(A\phi SY,N) + \beta g(\xi,JAN)q(Y) - \beta g(\xi,ASY) = 0. \tag{4.4}$$

On the other hand, we know that

$$\begin{aligned} g(\xi,ASY) &= -g(JN,ASY) = g(N,JASY) = -g(N,AJSY) \\ &= -g(N,A\phi SY) - \eta(SY)g(N,AN). \end{aligned}$$

Substituting this one into (4.4), we have

$$\beta g(\xi,JAN)q(Y) + \beta\eta(SY)g(N,AN) = 0.$$

From this, together with $g(AN,N) = -\beta$, we have

$$\beta^2\{q(Y) - \alpha\eta(Y)\} = 0.$$

Then $\beta = 0$ or $q(Y) = \alpha\eta(Y)$ for any vector field $Y$ on $M$ in $Q^m$.

When the function $\beta = g(A\xi,\xi) = 0$, we have $t = \frac{\pi}{4}$, because $\beta = -\cos 2t$ in section 3, then the unit normal vector field $N$ becomes

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$ as in section 3, that is, the unit normal $N$ is $\mathfrak{A}$-isotropic.

Next we consider the case that $\beta \neq 0$. Then $q(\xi) = \alpha$.

Now let us put $Y = \xi$ in (4.3) and use the assumption of $M$ being Hopf, that is, $S\xi = \alpha\xi$. Then we have

$$\begin{aligned} 0 = &(\xi\beta)g(AX,N) + \beta\{q(\xi)g(JAX,N) + \alpha\eta(X)g(AN,N)\} \\ &- q(\xi)g(X,A\xi)g(JA\xi,N) \\ &- \alpha g(X,A\xi)g(AN,N) - q(\xi)g(X,JAN)g(AN,N) + \alpha g(X,A\xi)g(AN,N) \\ &- q(\xi)g(JAN,N)g(X,AN). \end{aligned} \tag{4.5}$$

From this, by putting $q(\xi) = \alpha$, and using $g(AN,N) = -\beta$, $g(JA\xi,N) = -g(AN,N)$, and $g(JAN,N) = g(AN,\xi) = 0$, we have

$$\begin{aligned} 0 &= (\xi\beta)g(AX,N) + \alpha\beta g(JAX,N) - \alpha\beta^2\eta(X) \\ &\quad + \alpha\beta g(X,JAN) - \alpha\beta g(X,A\xi) \\ &= (\xi\beta)g(AX,N) + \alpha\beta g(AX,\xi) - \alpha\beta^2\eta(X), \end{aligned} \tag{4.6}$$

where we have used that $g(JAX, N) = g(A\xi, X)$ and $g(X, JAN) = g(X, A\xi)$. Here we know that $\xi\beta = 0$, because we can calculate the following

$$\begin{aligned}\xi\beta &= \xi g(A\xi, \xi)\\ &= g((\bar{\nabla}_\xi A)\xi + A\bar{\nabla}_\xi \xi, \xi) + g(A\xi, \bar{\nabla}_\xi \xi)\\ &= g(q(\xi)JA\xi, \xi)\\ &= -q(\xi)g(A\xi, N)\\ &= 0,\end{aligned} \tag{4.7}$$

where we have used the equation of Gauss $\bar{\nabla}_\xi \xi = \nabla_\xi \xi + g(S\xi, \xi)N = \alpha N$. Then (4.6) gives $0 = \alpha\beta g(A\xi, X) - \alpha\beta^2 g(\xi, X)$ for any tangent vector field $X$ on $M$. From this, together with $0 = \alpha\beta g(A\xi, N) - \alpha\beta^2 g(\xi, N)$ for the unit normal vector field $N$, we have

$$\alpha\beta A\xi = \alpha\beta^2 \xi. \tag{4.8}$$

By applying the complex conjugation $A$ to both sides of (4.8) and using the involution property $A^2 = I$ and (4.8) again, we get

$$\alpha\beta\xi = \alpha\beta A^2 \xi = \alpha\beta^2 A\xi = \alpha\beta^3 \xi.$$

From this, together with using the property of $\alpha\beta \neq 0$ for the Reeb function $\alpha \neq 0$, we have $\beta^2 = 1$. This means that $\beta = -\cos 2t = 1$ or $\beta = -\cos 2t = -1$ if the Reeb function $\alpha$ is non-vanishing, because the function $\beta = g(A\xi, \xi) = -\cos 2t$ as in section 3. Then we have respectively $t = \frac{\pi}{2}$ or $t = 0$. But in section 3, we know that $0 \leq t \leq \frac{\pi}{4}$. So we have only $t = 0$, and the unit normal vector field $N$ becomes $\mathfrak{A}$-principal. Then including the case of vanishing Reeb curvature $\alpha$, we can prove the following

**Lemma 4.1.** *Let $M$ be a Hopf real hypersurface in complex quadric $Q^m$, $m \geq 3$, with parallel structure Jacobi operator. Then the unit normal vector field $N$ is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

**Proof.** When the Reeb function $\alpha$ is non-vanishing, we have shown that the unit normal $N$ is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal according to the function $\beta = 0$ or $\beta = -1$ respectively. When the Reeb function $\alpha$ identically vanishes, let us show that $N$ is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. In order to do this, from the condition of Hopf, we can differentiate $S\xi = \alpha\xi$ and use the equation of Codazzi in section 3, then we get the formula

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

From the assumption of $\alpha = 0$ combined with the fact $g(\xi, AN) = 0$ proved in section 3, we deduce $g(Y, AN)g(\xi, A\xi) = 0$ for any $Y \in T_z M$, $z \in M$. This gives that the vector $AN$ is normal, that is, $AN = g(AN, N)N$ or $g(A\xi, \xi) = 0$, which implies that the unit normal $N$ is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic, respectively. This completes the proof of our Lemma. □

By virtue of Lemma 4.1, we can consider two classes of real hypersurfaces in complex quadric $Q^m$ with parallel structure Jacobi operator with $\mathfrak{A}$-principal unit normal vector field $N$ or otherwise, with $\mathfrak{A}$-isotropic unit normal vector field $N$. We will consider each cases in sections 5 and 6 respectively.

## 5. Parallel structure Jacobi operator with $\mathfrak{A}$-principal normal

In this section we consider a real hypersurface $M$ in a complex quadric with $\mathfrak{A}$-principal unit normal vector field. Then the unit normal vector field $N$ satisfies $AN = N$ for a complex conjugation $A \in \mathfrak{A}$.

Then the structure Jacobi operator $R_\xi$ is given by

$$R_\xi(X) = X - 2\eta(X)\xi - AX + g(S\xi,\xi)SX - g(SX,\xi)S\xi. \tag{5.1}$$

Since we assume that $M$ is being Hopf, (5.1) becomes

$$R_\xi(X) = X - 2\eta(X)\xi - AX + \alpha SX - \alpha^2\eta(X)\xi. \tag{5.2}$$

By the assumption of the structure Jacobi operator $R_\xi$ being parallel, the derivative of $R_\xi$ along any tangent vector field $Y$ on $M$ is given by

$$\begin{aligned} 0 &= (\nabla_Y R_\xi)(X) = \nabla_Y(R_\xi(X)) - R_\xi(\nabla_Y X) \\ &= -2\{(\nabla_Y\eta)(X)\xi + \eta(X)\nabla_Y\xi\} - (\nabla_Y A)X + (Y\alpha)SX \\ &\quad + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi - \alpha^2(\nabla_Y\eta)(X)\xi - \alpha^2\eta(X)\nabla_Y\xi. \end{aligned} \tag{5.3}$$

We can write

$$AY = BY + \rho(Y)N,$$

where $BY$ denotes the tangential component of $AY$ and $\rho(Y) = g(AY, N) = g(Y, AN) = g(Y, N) = 0$. So we have $AY = BY$ for any vector field $Y$ on $M$ in $Q^m$. Then it follows

$$\begin{aligned} (\nabla_Y A)X &= \nabla_Y(AX) - A\nabla_Y X \\ &= \bar{\nabla}_Y(AX) - \sigma(Y, AX) - A\nabla_Y X \\ &= (\bar{\nabla}_Y A)X + A\{\nabla_Y X + \sigma(Y, X)\} \\ &\quad - \sigma(Y, AX) - A\nabla_Y X \\ &= q(Y)JAX + A\sigma(Y, X) - \sigma(Y, AX) \\ &= q(Y)JAX + g(SX, Y)AN - g(SY, AX)N, \end{aligned} \tag{5.4}$$

where we have used the Gauss and Weingarten formulae. From this, together with (5.3) and using the notion of $\mathfrak{A}$-principal, we have

$$\begin{aligned} 0 &= (\nabla_Y R_\xi)(X) \\ &= -(2 + \alpha^2)\{(\nabla_Y\eta)(X)\xi + \eta(X)\nabla_Y\xi\} \\ &\quad - \{q(Y)JAX + g(SX, Y)N - g(SY, AX)N\} \\ &\quad + (Y\alpha)SX + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi. \end{aligned} \tag{5.5}$$

From this, taking the inner product of (5.5) with the unit normal vector field $N$, we have

$$q(Y)g(JAX, N) + g(SX, Y) - g(SY, AX) = 0.$$

Since $A\xi = -\xi$, the formula $g(JAX, N) = g(AX, \xi) = -\eta(X)$ holds. Then we have

$$-q(Y)\xi + SY - ASY = 0.$$

By putting $Y = \xi$ and using the assumption of $M$ being Hopf, we have

$$q(\xi) = 2\alpha. \tag{5.6}$$

Putting $X = \xi$ into (5.5), and using (5.6) and the constancy of the Reeb function $\alpha = g(S\xi, \xi)$ (see Lemma 3.2 in section 3), we have

$$\begin{aligned} 0 =& -(2+\alpha^2)\nabla_Y\xi \\ &- \{2\alpha\eta(Y)JA\xi + 2\alpha\eta(Y)N\} + \alpha(\nabla_Y S)\xi \\ =& -2\phi SY - \alpha S\phi SY, \end{aligned} \tag{5.7}$$

where we have used

$$\begin{aligned} (\nabla_Y S)\xi =& \nabla_Y(S\xi) - S\nabla_Y\xi \\ =& \alpha\nabla_Y\xi - S\phi SY \\ =& \alpha\phi SY - S\phi SY. \end{aligned} \tag{5.8}$$

If we put $SY = \lambda Y$, $Y \in \mathcal{C} = [\xi]^\perp$, where $Y$ is orthogonal to the Reeb vector field $\xi$, then (5.7) gives

$$2\lambda\phi Y = -\alpha\lambda S\phi Y. \tag{5.9}$$

Here we can show that the principal curvature $\lambda$ identically vanishes on $M$. In fact, if we assume that there is a principal curvature vector field $Y \in \mathcal{C}$ such that $SY = \lambda Y$, $\lambda \neq 0$, then (5.9) yields

$$S\phi Y = -\frac{2}{\alpha}\phi Y. \tag{5.10}$$

But by Lemma 3.2, we know that $S\phi Y = \mu\phi Y$, $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$ for $SY = \lambda Y$. From this, together with (5.10), it follows that $\alpha^2 + 4 = 0$, which gives a contradiction. Then the expression of the shape operator $S$ of $M$ in $Q^m$ satisfies

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

This gives $SY = \alpha\eta(Y)\xi$ for any tangent vector field $Y$ on $M$, where $\eta$ is an 1-form corresponding to the Reeb vector field $\xi$, that is, $M$ is totally $\eta$-umbilical, in which case the shape operator $S$ commutes with the structure tensor $\phi$. Then by Theorem A in the introduction, $M$ is locally congruent to a tube of radius $r$, $0 < r < \frac{\pi}{2}$, over a totally geodesic complex submanifold $\mathbb{C}P^k$ in $Q^{2k}$, $m = 2k$. That is, the Reeb flow on $M$ is isometric.

On the other hand, we want to introduce the following proposition (see page 1350050-14 in Berndt and Suh [3]).

**Proposition 5.1.** *Let $M$ be a real hypersurface in $Q^m$, $m\geq 3$, with isometric Reeb flow. Then the unit normal vector field $N$ is $\mathfrak{A}$-isotropic everywhere.*

By Proposition 5.1, we know that the unit normal vector field $N$ of $M$ is $\mathfrak{A}$-isotropic, not $\mathfrak{A}$-principal. This rules out the existence of a real hypersurface in $Q^m$, $m\geq 3$, with parallel structure Jacobi field and $\mathfrak{A}$-principal unit normal vector field $N$. Accordingly, such an $\mathfrak{A}$-principal case for parallel structure Jacobi operator on the tube never happen. So we give a proof of our main theorem with $\mathfrak{A}$-principal unit normal $N$.

## 6. Parallel structure Jacobi operator with $\mathfrak{A}$-isotropic normal

In this section we assume that the unit normal vector field $N$ is $\mathfrak{A}$-isotropic. Then the normal vector field $N$ can be written as

$$N = \frac{1}{\sqrt{2}}(Z_1 + JZ_2)$$

for $Z_1, Z_2 \in V(A)$, where $V(A)$ denotes a $+1$-eigenspace of the complex conjugation $A\in\mathfrak{A}$. Then it follows that

$$AN = \frac{1}{\sqrt{2}}(Z_1 - JZ_2),\ AJN = -\frac{1}{\sqrt{2}}(JZ_1 + Z_2),\ \text{and}\ JN = \frac{1}{\sqrt{2}}(JZ_1 - Z_2).$$

Then it gives that

$$g(\xi, A\xi) = g(JN, AJN) = 0, g(\xi, AN) = 0 \text{ and } g(AN, N) = 0.$$

By virtue of these formulas for $\mathfrak{A}$-isotropic unit normal, the structure Jacobi operator can be defined so that

$$\begin{aligned} R_\xi(X) &= R(X,\xi)\xi \\ &= X - \eta(X)\xi - g(AX,\xi)A\xi - g(JAX,\xi)JA\xi \\ &\quad + g(S\xi,\xi)SX - g(SX,\xi)S\xi. \end{aligned} \tag{6.1}$$

On the other hand, we know that $JA\xi = -JAJN = AJ^2N = -JN$, and $g(JAX,\xi) = -g(AX,J\xi) = -g(AX,N)$. Now the structure Jacobi operator $R_\xi$ can be rearranged as follows:

$$\begin{aligned} R_\xi(X) &= X - \eta(X)\xi - g(AX,\xi)A\xi - g(X,AN)AN \\ &\quad + \alpha SX - \alpha^2\eta(X)\xi. \end{aligned} \tag{6.2}$$

Differentiating (6.2) we obtain

$$\begin{aligned} \nabla_Y R_\xi(X) &= \nabla_Y(R_\xi(X)) - R_\xi(\nabla_Y X) \\ &= -(\nabla_Y\eta)(X)\xi - \eta(X)\nabla_Y\xi - g(X,\nabla_Y(A\xi))A\xi \\ &\quad - g(X,A\xi)\nabla_Y(A\xi) - g(X,\nabla_Y(AN))AN - g(X,AN)\nabla_Y(AN) \end{aligned} \tag{6.3}$$

$$
\begin{aligned}
&+ (Y\alpha)SX + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi \\
&- \alpha^2(\nabla_Y \eta)(X)\xi - \alpha^2\eta(X)\nabla_Y\xi.
\end{aligned}
$$

Here let us use the equation of Gauss and Weingarten formula as follows:

$$
\begin{aligned}
\nabla_Y(A\xi) &= \bar{\nabla}_Y(A\xi) - \sigma(Y, A\xi) \\
&= (\bar{\nabla}_Y A)\xi + A\bar{\nabla}_Y\xi - \sigma(Y, A\xi) \\
&= q(Y)JA\xi + A\{\phi SY + \eta(SY)N\} - g(SY, A\xi)N,
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_Y(AN) &= \bar{\nabla}_Y(AN) - \sigma(Y, AN) \\
&= (\bar{\nabla}_Y A)N + A\bar{\nabla}_Y N - \sigma(Y, AN) \\
&= q(Y)JAN - ASY - g(SY, AN)N.
\end{aligned}
$$

Substituting these formulas into (6.3) and using the assumption of parallel structure Jacobi operator, we have

$$
\begin{aligned}
0 &= \nabla_Y R_\xi(X) \\
&= -g(\phi SY, X)\xi - \eta(X)\phi SY \\
&\quad - \{q(Y)g(A\xi, X) + g(A\phi SY, X) + g(SY, \xi)g(AN, X)\}A\xi \\
&\quad - g(X, A\xi)\{q(Y)JA\xi + A\phi SY + g(SY, \xi)AN \\
&\quad - g(SY, A\xi)N\} - \{q(Y)g(X, AN) - g(X, ASY)\}AN \\
&\quad - g(X, AN)\{q(Y)JAN - ASY - g(SY, AN)N\} \\
&\quad + (Y\alpha)SX + \alpha(\nabla_Y S)X - (Y\alpha^2)\eta(X)\xi \\
&\quad - \alpha^2 g(\phi SY, X)\xi - \alpha^2\eta(X)\phi SY.
\end{aligned}
\tag{6.4}
$$

From this, taking inner product with the Reeb vector field $\xi$, we have

$$
\begin{aligned}
0 &= -g(\phi SY, X) - g(X, A\xi)g(A\phi SY, \xi) + g(X, AN)g(ASY, \xi) \\
&\quad + (Y\alpha)\alpha\eta(X) + \alpha g((\nabla_Y S)X, \xi) \\
&\quad - (Y\alpha^2)\eta(X) - \alpha^2 g(\phi SY, X).
\end{aligned}
\tag{6.5}
$$

Here by the assumption of $M$ being Hopf, we can use the following

$$
(\nabla_Y S)\xi = \nabla_Y(S\xi) - S(\nabla_Y\xi) = (Y\alpha)\xi + \alpha\phi SY - S\phi SY.
$$

Then it follows that

$$
\alpha g((\nabla_Y S)X, \xi) = g(\alpha(Y\alpha)\xi + \alpha^2\phi SY - \alpha S\phi SY, X). \tag{6.6}
$$

Taking inner product of (6.4) with the unit normal $N$, it follows that

$$\begin{aligned} 0 &= -g(X, A\xi)g(A\phi SY, N) + g(X, A\xi)g(SY, A\xi) \\ &\quad + g(X, AN)g(ASY, N) + g(X, AN)g(SY, AN). \end{aligned} \tag{6.7}$$

From this, putting $X = AN$ and using that $N$ is $\mathfrak{A}$-isotropic, we have $SAN = 0$. This also gives $S\phi A\xi = 0$.

On the other hand, one of the terms $g(SY, A\xi)$ in (6.4) becomes

$$g(SY, A\xi) = -g(SY, AJN) = g(SY, JAN) = g(SY, \phi AN + \eta(AN)N) = -g(A\phi SY, N).$$

Substituting this term into (6.7) gives $S\phi AN = 0$. Summing up these formulas, we can write

$$SA\xi = 0,\ \ SAN = 0,\ \ S\phi A\xi = 0, \text{and } S\phi AN = 0. \tag{6.8}$$

Taking the inner product of (6.4) with the Reeb vector field $\xi$, and using (6.6), (6.8) and the constancy of the Reeb function $\alpha$ in Lemma 3.3, we have

$$\phi SY = -\alpha S\phi SY. \tag{6.9}$$

In the case for $N$ is $\mathfrak{A}$-isotropic, in Lemma 3.3 we have shown that the Reeb function $\alpha$ is constant. So, we divide into the two cases that either $\alpha = 0$ or $\alpha \neq 0$. For the first case with the Reeb function $\alpha$ vanishing, (6.9) gives $\phi SY = 0$, which implies $SY = \alpha\eta(Y)\xi = 0$ for any vector field $Y$ on $M$, that is, $M$ is totally geodesic. Then by putting $X = \xi$ into the equation of Codazzi in section 3 for $\mathfrak{A}$-isotropic unit normal vector field $N$ and using the notion of totally geodesic, we have

$$0 = g(\phi Y, Z) - g(Y, AN)g(A\xi, Z) - g(Y, A\xi)g(JA\xi, Z).$$

Then for any vector fields $Y, Z \in \mathcal{Q}$, where $Y, Z$ are orthogonal to the Reeb vector fields $A\xi$ and $AN$, we have $g(\phi Y, Z) = 0$, which gives a contradiction.

Next we consider the case for the Reeb function $\alpha \neq 0$.

On the distribution $\mathcal{Q}$ let us introduce a formula mentioned in section 3 as follows:

$$2S\phi SY - \alpha(\phi S + S\phi)Y = 2\phi Y \tag{6.10}$$

for any tangent vector field $Y$ on $M$ in $Q^m$ (see also [3], pages 1350050-11). So if $SY = \lambda Y$ in (6.10), then $(2\lambda - \alpha)S\phi Y = (\alpha\lambda + 2)\phi Y$, which gives

$$S\phi Y = \frac{\alpha\lambda + 2}{2\lambda - \alpha}\phi Y. \tag{6.11}$$

Here we note that $2\lambda - \alpha \neq 0$. In fact, if $2\lambda - \alpha = 0$, then $\alpha\lambda + 2 = 0$, which implies $\alpha^2 + 4 = 0$. This gives us a contradiction. By (6.9) and (6.10), we know that

$$-\frac{2 + \alpha^2}{\alpha}\phi SY - \alpha S\phi Y = 2\phi Y.$$

From this, putting $SY = \lambda Y$ and using (6.11), we know that

$$\begin{aligned} S\phi Y &= -\frac{2\lambda + \alpha^2\lambda + 2\alpha}{\alpha^2}\phi Y \\ &= \frac{\alpha\lambda + 2}{2\lambda - \alpha}\phi Y. \end{aligned} \tag{6.12}$$

Then by a straightforward calculation, we get the following equation

$$\lambda\{2(\alpha^2+2)\lambda+2\alpha\}=0.$$

This means $\lambda=0$ or $\lambda=-\frac{\alpha}{\alpha^2+2}$. When $\lambda=0$, by (6.12), $S\phi Y=-\frac{2}{\alpha}\phi Y$. Then $\frac{2}{\alpha}=\frac{\alpha}{\alpha^2+2}$, which gives $\alpha^2+4=0$. This is again a contradiction. So we can assume that the other principal curvature is $-\frac{\alpha}{\alpha^2+2}$. Now let us denote the principal curvature $-\frac{\alpha}{\alpha^2+2}$ by the function $\beta$. Accordingly, the shape operator $S$ can be expressed as

$$S=\begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \beta & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \beta & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \beta \end{bmatrix}$$

Let us consider the principal curvature $\beta$ such that $SY=\beta Y$ in the formula (6.9). Then (6.9) gives that $\beta\phi Y=-\alpha\beta S\phi Y$. From this, together with the expression for $S$, we have

$$\begin{aligned} S\phi Y &= \beta\phi Y \\ &= -\frac{\beta}{\alpha\beta}\phi Y=-\frac{1}{\alpha}\phi Y. \end{aligned}$$

Then $-1=\alpha\beta=-\frac{\alpha^2}{\alpha^2+2}$, which gives us a contradiction. Accordingly, we also conclude that any real hypersurfaces $M$ in $Q^m$ with $\mathfrak{A}$-isotropic unit normal vector field and the non-vanishing Reeb function $\alpha$ do not admit a parallel structure Jacobi operator.

**Remark 6.1.** In [19] we have classified real hypersurfaces $M$ in complex quadric $Q^m$ with parallel Ricci tensor, according to whether the unit normal $N$ is $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. When $N$ is $\mathfrak{A}$-principal, we proved a non-existence property for Hopf hypersurfaces in $Q^m$. For a Hopf real hypersurface $M$ in $Q^m$ with $\mathfrak{A}$-isotropic we have given a complete classification that it has *three distinct constant* principal curvatures.

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